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Note on the Hydrodynamic Eigenmodes of Couette Flow

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The axisymmetric eigenmodes for the velocity and pressure disturbances in the incompressible Couette flow between two concentric rotating cylinders with no-slip boundary conditions are computed numerically and plotted. As found previously in the narrow-gap approximation as well as for the Rayleigh–Bénard system, damped propagating viscous modes are present for wide ranges of parameters. Closed-form solutions for the special case of corotating cylinders show that the time constants for the decay of the eigenmodes then become insensitive to the ratio of the radii of the cylinders.

KEY WORDS: Circular Couette flow; hydrodynamic eigenmodes; damped propagating modes.

The Couette system of fluid flow between two concentric rotating cylinders has been a paradigm for many theoretical and experimental investigations[1,2] on the subject of instability and bifurcation; in particular, the first transition from uniform circular flow to Taylor vortex flow has been extensively investigated.[3] We shall be concerned here, however, with small damped disturbances around the uniform circular flow, the Couette flow, before the onset of the transition to the Taylor vortex flow takes place.

One of the interesting effects occurring in the stable regime of a fluid subject to a constant temperature gradient, i.e., for the Rayleigh–Bénard system, is that if a stabilizing gradient is applied (heating from above), the heat and viscous modes—which are purely diffuse in equilibrium—couple and become propagating for sufficiently large negative Rayleigh numbers. This effect has been measured in an experiment of forced Rayleigh scattering by Boon *et al.*[4,5] Since the governing equations for these visco-heat

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modes for the Rayleigh-Bénard system are similar to the equations for the viscous modes of the Couette system, Cohen and Schmitz[6] suggested that a similar propagation might also be present in the Couette system. Following this suggestion, we computed earlier the eigenvalue spectra for the viscous modes of the Couette system in the narrow-gap approximation[7] and found indeed propagating modes in a wide range of system parameters. In fact, since the narrow-gap equations are identical with those for the Rayleigh–Bénard system for special values of parameters, the eigenvalue spectra in the two systems are similar near those parameter values. The Couette system, however, has more parameters and hence shows a richer behavior. In this note, we extend our previous computations to the general case of arbitrary gap size.

We consider the Couette system consisting of two concentric cylinders with inner and outer radii R_1 and R_2 which rotate with angular velocities Ω_1 and Ω_2, respectively. The motion of the fluid, with kinematic viscosity ν, between the cylinders is uniformly circular. The eigenmodes are described by the incompressible Navier–Stokes equations, linearized around circular Couette flow. This yields for small velocity disturbances an eigenvalue problem for the three components of the velocity disturbance and the accompanying pressure disturbance. Restricting consideration to the axisymmetric modes and ignoring the boundaries at the top and bottom of the cylinders, we have that an eigenmode is characterized by a time constant s, an axial wavenumber k, and a radial eigenfunction for each velocity component since the eigenmodes can be expressed in the form

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \\ P \end{pmatrix} = \begin{pmatrix} u(r)/c_1 \\ v(r) \\ iw(r)/c_1 \\ p(r)/c_2 \end{pmatrix} e^{-st+ikz} \tag{1}$$

where u_r, u_θ, and u_z are the components of the velocity disturbance in cylindrical coordinates, P is the pressure disturbance divided by the density, u, v, w, p are the eigenfunctions, and c_1, c_2 are constants.

In general, the system of equations is not solvable in closed form, so we follow the method of Chandrasekhar[2]: First the full system of equations is simplified by eliminating P and u_z. Using the shorthand notation $D = d_\zeta$ and $D_* = d_\zeta + 1/\zeta$ with $\zeta = r/R_2$ and the dimensionless constants $a = kR_2$ and $\sigma = sR_2^2/\nu$, as well as setting $c_1 = 2\Omega_1(\mu - \eta^2)\, R_2^2/\nu(1-\eta^2)$ and $c_2 = c_1 R_2/\nu$, one obtains the familiar equations

$$\begin{gathered}(DD_* - a^2 + \sigma)(DD_* - a^2)\, u(\zeta) = Ta^2(\kappa - \zeta^{-2})\, v(\zeta) \\ (DD_* - a^2 + \sigma)\, v(\zeta) = u(\zeta)\end{gathered} \tag{2}$$

where the dimensionless constants are angular velocity ratio $\mu = \Omega_2/\Omega_1$, radius ratio $\eta = R_1/R_2$, Taylor number

$$T = 4\Omega_1^2 R_1^4 (1-\mu)(1-\mu/\eta^2)(1-\eta^2)^{-2} \nu^{-2}$$

and $\kappa = (1-\mu/\eta^2)/(1-\mu)$. The pressure and axial velocity components are then obtained from the radial velocity component using $ap(\zeta) = (D_* D - a^2 + \sigma)\, w(\zeta)$ and $aw(\zeta) = D_* u(\zeta)$. We use no-slip boundary conditions (b.c.), i.e., the fluid sticks to the side walls of the cylinders, which are $v(\zeta) = u(\zeta) = D_* u(\zeta) = 0$ for $\zeta = 1$ and η. The method of solution is a variational method in which $u(\zeta)$ and $v(\zeta)$ are expanded in a complete set of orthogonal functions with coefficients to be determined by the secular equation. This is done numerically with truncated expansions; the method requires only the first few terms of the orthogonal functions to achieve a high degree of convergence.

The results are shown in Figs. 1 and 2 for three radius ratios for two values of μ, one for rotation of the cylinders in the same direction and the other for rotation in opposite directions. It is more illuminating to present them in dimensionless parameters using the gap width $d = R_2 - R_1$, i.e., in $\tilde{a} = kd$, $\tilde{\sigma} = sd^2/\nu$, and $\tilde{T} = (1-\eta^2)(1-\eta)^4 \,|T|/\eta^2\, |1-\mu|$. Because $T\kappa \geqslant 0$, T and κ are both positive for $\mu > 1$ or $\mu < \eta^2$ and negative for $\eta^2 < \mu < 1$. Hence, the absolute sign in the definition of $\tilde{T}$ presents no ambiguity.

The spectra can be categorized in two regimes, in analogy to the

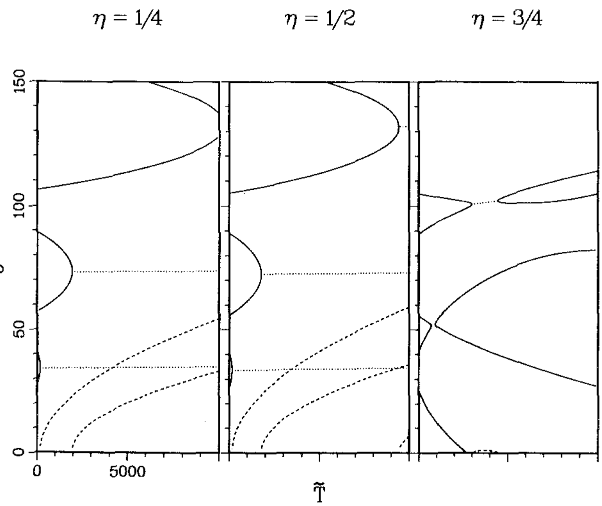

Fig. 1. Spectra for $\mu = 0.5$ and $\tilde{a} = 4$; solid lines are real eigenvalues and dotted (dashed) lines are the real (imaginary) parts of complex eigenvalues. The abscissas of the three plots have the same linear scale from 0 to 10,000.

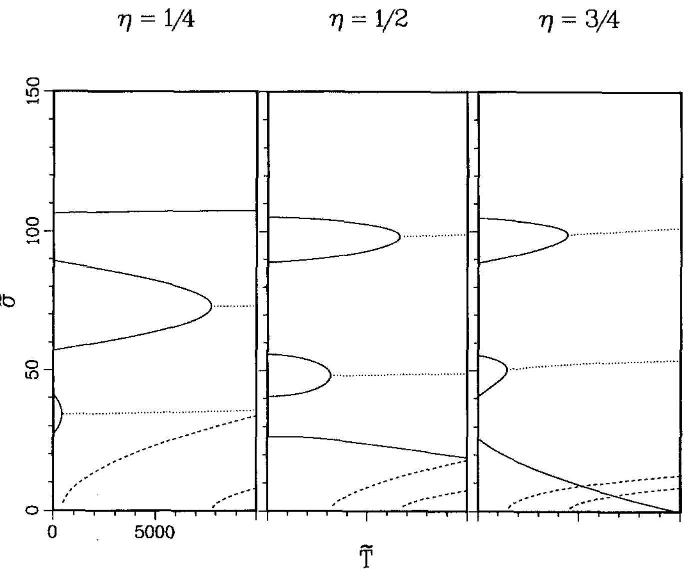

Fig. 2. Spectra for $\mu = -0.4$ and $\tilde{a} = 4$; see also caption of Fig. 1.

heating-from-above and the heating-from-below regimes of the Rayleigh–Bénard system; these regimes are determined by the nature of the fundamental mode, namely, in the first regime (analogous to heating from above) this mode exhibits predominantly a damped propagation, while in the second regime it is purely diffusive, i.e., purely damped. The first two plots of Fig. 1 and the first plot of Fig. 2 are in the first regime; the eigenvalue of the fundamental mode is real only for very small values of $\tilde{T}$ and is represented by the lower portion of the lowest solid curve, while it is complex in the other regions of the plots, with its real part represented by the lowest dotted line and its imaginary part by the highest dashed line. The rest of the two figures is in the second regime; the eigenvalue of the fundamental mode is then real, represented by the lowest solid curve, which decreases with increasing Taylor number until it vanishes at some critical Taylor number T_c. At this point the transition to Taylor vortex flow takes place, so that the spectrum ceases to be meaningful beyond T_c, since the linearized equations are no longer valid. Propagation of the higher eigenmodes in the second regime occurs over a large range of Taylor numbers in the counterrotating case (Fig. 2), but not in the other case (Fig. 1). Since these modes are highly damped, they could only be detected, e.g., in the short-time behavior of the system, when the higher eigenmodes are still important. For the long-time behavior, the system is dominated by the lowest eigenmode, which propagates only in the first regime.

Comparing the above results with the corresponding spectra of the Rayleigh–Bénard system,[7,8] we note the following. The eigenvalue equations for the visco-heat modes in the Rayleigh–Bénard system at Prandtl number 1 are similar to those for the viscous modes in the Couette system, especially in the narrow-gap approximation; the Rayleigh number in the former plays the same role as the Taylor number in the latter.[6] For fixed system parameters in the Rayleigh–Bénard system, there is always a point of instability for sufficiently large positive Rayleigh numbers (negative temperature gradients) and a point of propagation for sufficiently large negative Rayleigh numbers. However, for fixed η and μ in the Couette system, the Taylor number is restricted to one sign and hence the system can either become strongly propagating or become unstable as the Taylor number is varied. As to the detailed behavior of the spectra, those of the Rayleigh–Bénard system as a function of the Rayleigh number can be approximated by a parabolic function for a wide range of Rayleigh numbers.[9] This feature is also present in the Couette system in Fig. 1, where $\mu > 0$, and in Fig. 2, where $\mu < 0$. We note that only the case $\mu > 0$ is analogous to the Rayleigh–Bénard system; for $\mu < 0$, i.e., counterrotating cylinders, the eigenvalue spectra are quite different for the two systems. Also, we found no analogy in the Rayleigh–Bénard system for the feature of wide regions of propagation in the higher modes appearing in these figures.

For the special case in which the two cylinders rotate with the same angular velocity, i.e., for $\mu = 1$, Eqs. (2) can be solved in closed form. This is because for $\mu \to 1$, $T \to 0$ and $\kappa \to \infty$, with $T\kappa = \tilde{T}/(1-\eta)^4$; hence the term containing $1/\zeta^2$ vanishes and the equations contain only the Bessel operator DD_*. The eigenfunctions for the velocity disturbances are now expressible as linear combinations of Bessel functions:

$$
\begin{aligned}
v(\zeta) &= \sum_{k=1}^{3} [C_{2k-1} I_1(\beta_k \zeta) + C_{2k} K_1(\beta_k \zeta)] \\
u(\zeta) &= \sum_{k=1}^{3} (\beta_k^2 - a^2 + \sigma)[C_{2k-1} I_1(\beta_k \zeta) + C_{2k} K_1(\beta_k \zeta)] \qquad (3) \\
w(\zeta) &= \frac{1}{a} \sum_{k=1}^{3} \beta_k(\beta_k^2 - a^2 + \sigma)[C_{2k-1} I_0(\beta_k \zeta) - C_{2k} K_0(\beta_k \zeta)]
\end{aligned}
$$

where $I_n(z)$ and $K_n(z)$ are, respectively, the nth-order modified Bessel functions of the first and second kinds and the coefficients C_j, $j = 1,..., 6$, are determined by the b.c. The constants β_l^2, $l = 1, 2, 3$, in the argument of the Bessel functions are the roots of a cubic equation,

$(\beta^2 - a^2 + \sigma)^2 (a^2 - \beta^2) + T\kappa a^2 = 0$, in terms of the eigenvalue σ, which is then determined by the b.c.

$$v(1) = v(\eta) = u(1) = u(\eta) = w(1) = w(\eta) = 0$$

This yields a set of linear homogeneous equations for the C_j, $j = 1,..., 6$, and nontriviality requires that the determinant of a 6×6 coefficient matrix vanishes. Some results are shown in Fig. 3; this case is in the same regime as that of the first two plots of Fig. 1 and the spectra resemble those plots. The $\mu = 1$ case has the striking feature of being remarkably insensitive to variations in the radius ratio η. In addition, we also computed the eigenfunctions for the fundamental mode at $\tilde{T} = 5000$ and $\eta = 1/4$. The results are plotted in Fig. 4. We remark that each component is either predominantly real or predominantly imaginary. We have used the relation $\tilde{T} = (1 - \eta)^4 c_1^2$ to determine c_1, which is necessary for obtaining the correct relative magnitudes of the components of the velocity disturbance in these figures.

We note that propagating modes are found in the Rayleigh–Bénard system in the higher excited modes in a small parameter range, so-called "windows of propagation,"[8] which can be thought of as resulting from the interaction of two adjacent modes; the same feature is found in the spectra of the Couette system,[7] except that here the "interaction" can be so large that the eigenmodes are propagating in a wide region of parameters.

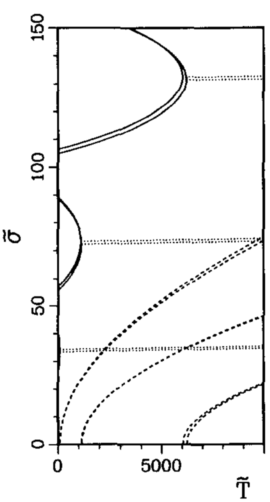

Fig. 3. Spectra for $\mu = 1$ and $\tilde{a} = 4$. The cases $\eta = 1/4$ and $\eta = 3/4$ are plotted; the latter has a slightly larger imaginary part of the complex eigenvalue, and a slightly smaller real eigenvalue or real part of the complex eigenvalues. See also caption of Fig. 1.

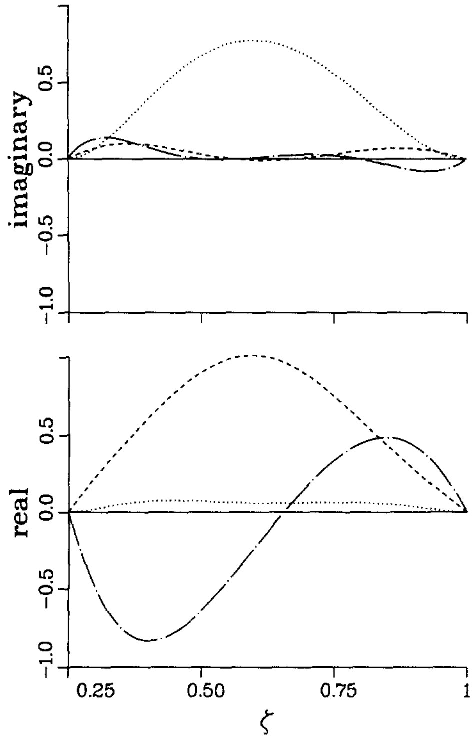

Fig. 4. The eigenfunctions of the fundamental mode for $\mu = 1$, $\eta = 1/4$, and $\tilde{T} = 5000$; the dotted, dashed, and chain-dashed lines describe $u(\zeta)/c_1$, $v(\zeta)$, and $w(\zeta)/c_1$, respectively.

The frequency of oscillation of a column of rotating water has been measured by Fultz[10]; he used a vertically oscillating small disk to excite particular modes, and observed the motion of some injected dye to determine the resonance frequencies. Despite the rather large amplitude of the motion of the disk, he found remarkable agreement with the predictions of the theory of inviscid fluids applied to small disturbances.

In addition to the damped propagating axisymmetric eigenmodes determined in this note, also spiral eigenmodes could be considered. In fact, for sufficiently large $T > T_c$ and $\eta < 0$, i.e., counterrotating cylinders, propagating spiral modes have been observed, so that damped propagating spiral modes could be expected for $T < T_c$.

Finally we remark that a simplification of the problem could be made by replacing the no-slip b.c. with simpler b.c. which are in between the two extremes of slip and no-slip b.c. Such b.c. could be "vertical-slip" b.c., which would allow only the axial component of the velocity disturbance to be nonvanishing at the walls. Then, instead of the no-slip b.c., we would have $v(\zeta) = u(\zeta) = DD_* u(\zeta) = 0$ for $\zeta = 1$ and η. These b.c. could be useful since results could be obtained much more easily than for no-slip b.c. In fact, using these b.c., one can make an estimate, for example, of the importance of the damping as compared to the oscillation in the Fultz experiment. For the parameters used in Fultz' experiment one finds then for the fundamental mode that the oscillation is precisely that of the inviscid theory, while the damping is 0.5% of the oscillation, in agreement with Fultz' observations. However, this same procedure could give estimates for the damping and oscillation for different parameter ranges as well.

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REFERENCES

1. H. L. Swinney and J. P. Gollub, eds., *Hydrodynamic Instabilities and the Transition to Turbulence* (Springer, Berlin, 1981), Chapter 6.
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1961), Chapter 7.
3. W. F. Langford, R. Tagg, E. J. Kostelich, H. L. Swinney, and M. Golubitsky, *Phys. Fluids* **31**:776 (1988); R. Tagg, W. S. Edwards, and H. L. Swinney, *Phys. Rev. A* **42**:831 (1990).
4. B. J. Berne and R. Pecora, *Dynamic Light Scattering* (Wiley, New York, 1976).
5. J. P. Boon, C. Allain, and P. Lallemand, *Phys. Rev. Lett.* **43**:199 (1979); C. Allain, P. Lallemand, and J. P. Boon, in *Light Scattering in Liquids and Macromolecular Solutions*, V. Degiorgio, M. Corti, and M. Giglio, eds. (Plenum Press, New York, 1980).
6. E. G. D. Cohen and R. Schmitz, in *Fundamental Problems in Statistical Mechanics*, Vol. 6, E. G. D. Cohen, ed. (North-Holland, Amsterdam, 1985), p. 229.
7. L. F. Ko and E. G. D. Cohen, *Phys. Lett. A* **125**:231 (1987).
8. R. Schmitz and E. G. D. Cohen, *J. Stat. Phys.* **39**:285 (1985); **40**:431 (1985); **46**:319 (1987).
9. H. W. Lekkerkerker and J. P. Boon, *Phys. Rev. A* **10**:1355 (1974).
10. D. Fultz, *J. Meteorol.* **16**:199 (1959).

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